

On q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker function I

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Abstract. We propose new explicit form of q -deformed Whittaker functions solving q -deformed $\mathfrak{gl}_{\ell+1}$ -Toda chains. In the limit $q \rightarrow 1$ constructed solutions reduce to classical class one $\mathfrak{gl}_{\ell+1}$ -Whittaker functions in the form proposed by Givental. An important property of the proposed expression for the q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker function is that it can be represented as a character of $\mathbb{C}^* \times GL(\ell+1)$. This provides a q -version of the Shintani-Casselman-Shalika formula for p -adic Whittaker function. The Shintani-Casselman-Shalika formula is recovered in the limit $q \rightarrow 0$ when the q -deformed Whittaker function is reduced to a character of a finite-dimensional representation of $\mathfrak{gl}_{\ell+1}$ expressed through Gelfand-Zetlin bases.

Introduction

Whittaker functions corresponding to semisimple finite-dimensional Lie algebras arise in various parts of modern mathematics. In particular, these functions appear in representation theory as matrix elements of infinite-dimensional representations, in the theory of quantum integrable systems as common eigenfunction of Toda chain quantum Hamiltonians, in string theory as generating functions of correlators in Type A topological string theory on flag manifolds and in number theory in a description of local Archimedean L -factors corresponding to automorphic representations. Although much studied, Whittaker functions seems have some deep properties that are not yet fully revealed.

In this paper we study the q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker functions. The q -deformed Whittaker function can be identified with a common eigenfunction of a set of commuting q -deformed Toda chain Hamiltonians. This q -deformed Toda chain (also known as the relativistic Toda chain [Ru]) was discussed in terms of representation theory of quantum groups in [Se1], [Et], [Se2] and an integral representation for the q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker function was constructed in [KLS]. Recently the q -deformed Toda chain attracts special interest due to its connection with quantum K -theory of flag manifolds [GiL]. In this paper we pursue another direction. Our principal motivation to study q -deformed Whittaker functions is that in this, more general setting, some important hidden properties of classical Whittaker functions become visible.

The main result of the paper is given by Theorem 2.1 where a new expression for the q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker function (for $q < 1$) is introduced. As a simple corollary of Theorem 2.1, the q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker function can be represented as a character of $\mathbb{C}^* \times GL(\ell+1)$. In the limit $q \rightarrow 1$ this leads to a similar representation of classical $\mathfrak{gl}_{\ell+1}$ -Whittaker function. This representation is not easy to perceive looking directly at the classical Whittaker functions. The importance of this representation of (q -deformed) $\mathfrak{gl}_{\ell+1}$ -Whittaker function becomes obvious if we notice that in the limit $q \rightarrow 0$ the constructed q -deformed Whittaker function reduces to p -adic Whittaker function. In this limit the representation as a character reduces to well-known Shintani-Casselman-Shalika representation of p -adic $GL_{\ell+1}$ -Whittaker function as a character of a

finite-dimensional representation of $GL_{\ell+1}$ [Sh],[CS]. Thus the representation of (q -deformed) $\mathfrak{gl}_{\ell+1}$ -Whittaker function as a character can be considered as a q -version of Shintani-Casselman-Shalika representation. Indeed, the constructed q -deformed Whittaker function is equal to zero outside a dominant weight cone of $\mathfrak{gl}_{\ell+1}$ similarly to the Shintani-Casselman-Shalika p -adic Whittaker function.

We expect that the representation of the classical Whittaker function as a character should provide important insights into the arithmetic geometry at an infinite place of $\overline{\text{Spec}(\mathbb{Z})}$. Let us also remark that taking into account results [CS] one should expect that in the case of an arbitrary semisimple Lie algebra \mathfrak{g} , q -deformed \mathfrak{g} -Whittaker function should be given by a character of $\mathbb{C}^* \times {}^L G(\mathbb{C})$ where $\text{Lie}({}^L G) = {}^L \mathfrak{g}$ is a Langlands dual Lie algebra.

It is worth mentioning that the $q \rightarrow 1$ limit of the explicit expression of the q -deformed Whittaker function proposed in this paper reduces to the integral representations for classical Whittaker functions introduced by Givental [Gi],[GKLO]. We consider this as a sign of an “arithmetic nature” of this integral representation. On the other hand the explicit solution has an obvious relation with Gelfand-Zetlin parametrization of finite-dimensional representations of $\mathfrak{gl}_{\ell+1}$ (and precisely reproduces Gelfand-Zetlin form of characters of finite-dimensional representations in the limit $q \rightarrow 0$). This duality of Gelfand-Zetlin and Givental representations was already noticed in [GLO].

Let us comment on our approach to derivation of explicit expressions for q -deformed Whittaker functions. It is known [Et] that defining difference equations for Macdonald polynomials are transformed into q -deformed Toda chain eigenfunction equations in a certain limit. This is a simple generalization of the Inozemtsev limit [I] transforming Calogero-Sutherland integrable model into standard Toda chain. The other ingredient we use is a recursive construction of Macdonald polynomials (analogous to the recursive construction for (q -deformed) Toda chain eigenfunctions [KL1],[KLS]). We combine these results to obtain recursive expression for q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker functions satisfying q -deformed $\mathfrak{gl}_{\ell+1}$ -Toda chain eigenfunction equations.

The explicit form of the q -deformed Whittaker function implies various interesting interpretations. This includes connections with representation theory (via characters of Demazure modules), geometry of quiver varieties, quantum cohomology of flag manifolds and will be discussed elsewhere [GLO2].

Finally note that eigenfunctions of q -deformed Toda chain were discussed previously (e.g. [KLS],[GKL1],[BF] and [FFJMM]). The relation of these constructions with the one proposed in this paper is an interesting question which deserves further considerations.

The paper is organized as follows. In Section 1 we recall a systems of mutually commuting difference Macdonald-Ruijsenaars operators and recursive construction of their common eigenfunctions. In Section 2 we derive recursive expression for solutions of q -deformed $\mathfrak{gl}_{\ell+1}$ -Toda chain. In Section 3 various limiting cases elucidating the construction of the q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker functions are discussed. In Section 4 details of the proof of the Theorem 2.1 are given.

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1 Macdonald-Ruijsenaars difference operators

In this section we recall relevant facts from the theory of Macdonald polynomials (see e.g. [Mac],[Kir],[AOS]).

Consider symmetric polynomials in variables $(x_1, \dots, x_{\ell+1})$ over the field $\mathbb{Q}(q, t)$ of rational functions in q, t . Given a partition $\Lambda = (0 \leq \Lambda_1 \leq \Lambda_2 \leq \dots \leq \Lambda_{\ell+1})$, denote by the same symbol Λ the Young diagram containing $\ell + 1$ rows with Λ_k boxes in the k -th row; and the upper row having the maximal length $\Lambda_{\ell+1}$.

Let m_Λ and π_Λ be polynomial bases of the space of symmetric polynomials indexed by partitions Λ :

$$m_\Lambda = \sum_{\sigma \in \mathfrak{S}_{\ell+1}} x_{\sigma(1)}^{\Lambda_1} x_{\sigma(2)}^{\Lambda_2} \cdot \dots \cdot x_{\sigma(\ell+1)}^{\Lambda_{\ell+1}},$$

$$\pi_\Lambda = \pi_{\Lambda_1} \pi_{\Lambda_2} \cdot \dots \cdot \pi_{\Lambda_{\ell+1}}, \quad \pi_n = \sum_{k=1}^{\ell+1} x_k^n,$$

where $\mathfrak{S}_{\ell+1}$ is the permutation group. Define a scalar product $\langle \cdot, \cdot \rangle_{q,t}$ on the space of symmetric functions over $\mathbb{Q}(q, t)$ as follows

$$\langle \pi_\Lambda, \pi_{\Lambda'} \rangle_{q,t} = \delta_{\Lambda, \Lambda'} \cdot z_\Lambda(q, t),$$

where

$$z_\Lambda(q, t) = \prod_{n \geq 1} n^{m_n} m_n! \cdot \prod_{k=1}^N \frac{1 - q^{\Lambda_k}}{1 - t^{\Lambda_k}}, \quad m_n = |\{k \mid \Lambda_k = n\}|.$$

In the following we always imply $q < 1$.

Definition 1.1 *Macdonald polynomials $P_\Lambda^{\mathfrak{gl}_{\ell+1}} = P_\Lambda^{\mathfrak{gl}_{\ell+1}}(x; q, t)$ are symmetric polynomial function over $\mathbb{Q}(q, t)$ such that*

$$P_\Lambda^{\mathfrak{gl}_{\ell+1}} = m_\Lambda + \sum_{\Lambda' < \Lambda} u_{\Lambda \Lambda'} m_{\Lambda'},$$

with $u_{\Lambda \Lambda'} \in \mathbb{Q}(q, t)$, and for $\Lambda \neq \Lambda'$

$$\langle P_\Lambda^{\mathfrak{gl}_{\ell+1}}, P_{\Lambda'}^{\mathfrak{gl}_{\ell+1}} \rangle_{q,t} = 0.$$

Macdonald polynomials are eigenfunctions of a set of mutually commuting Macdonald-Ruijsenaars difference operators [Mac], [Ru]

$$H_r^{\mathfrak{gl}_{\ell+1}} = \sum_{I_r} t^{r(r-1)/2} \prod_{i \in I_r, j \notin I_r} \frac{tx_i - x_j}{x_i - x_j} \prod_{m \in I_r} T_m, \quad r = 1, \dots, \ell + 1, \quad (1.1)$$

where the sum is over ordered subsets

$$I_r = \{i_1 < i_2 < \dots < i_r\} \subset \{1, 2, \dots, \ell + 1\}.$$

The simplest operator of this kind is given by

$$H_1^{\mathfrak{gl}_{\ell+1}} = \sum_{i=1}^{\ell+1} \prod_{j, j \neq i} \frac{tx_i - x_j}{x_i - x_j} q^{x_i \partial_{x_i}}.$$

The eigenvalues of H_r are given by (see e.g. [EK])

$$H_r^{\mathfrak{gl}_{\ell+1}} P_\Lambda^{\mathfrak{gl}_{\ell+1}}(x; q, t) = c_\Lambda^r P_\Lambda^{\mathfrak{gl}_{\ell+1}}(x; q, t),$$

$$c_\Lambda^r = \chi_r(q^{\sum_{i=1}^{\ell+1} E_{ii} \Lambda_i} t^{\sum_{i=1}^{\ell+1} E_{i,i}(\ell+1-i)}) = \sum_{I_r} \prod_{i \in I_r} q^{\Lambda_i} t^{\ell+1-i},$$

where $E_{i,j}$ are standard generators of $\mathfrak{gl}_{\ell+1}$, $I_r = (i_1 < i_2 < \dots < i_r) \subset \{1, 2, \dots, \ell+1\}$ and $\chi_r(g)$ are the character of fundamental representations $V_r = \bigwedge^r \mathbb{C}^{\ell+1}$ of $\mathfrak{gl}_{\ell+1}$. In terms of a generating series

$$H^{\mathfrak{gl}_{\ell+1}}(\xi) = \sum_{r=0}^{\ell+1} \xi^{\ell+1-r} H_r^{\mathfrak{gl}_{\ell+1}}, \quad H_0 = 1$$

we have

$$H^{\mathfrak{gl}_{\ell+1}}(\xi) P_\Lambda^{\mathfrak{gl}_{\ell+1}}(x; q, t) = \prod_{i=1}^{\ell+1} (\xi + t^{\ell+1-i} q^{\Lambda_i}) P_\Lambda^{\mathfrak{gl}_{\ell+1}}(x; q, t).$$

Define the following scalar product on symmetric functions of $(\ell+1)$ -variables $x_1, \dots, x_{\ell+1}$

$$\langle f, g \rangle'_{q,t} = \frac{1}{(\ell+1)!} \oint_{x_1=0} \cdots \oint_{x_{\ell+1}=0} \prod_{i=1}^{\ell+1} \frac{dx_i}{2\pi i x_i} f(x^{-1}) g(x) \Delta(x|q, t),$$

where

$$\Delta(x|q, t) = \prod_{i \neq j} \prod_{n=0}^{\infty} \frac{1 - x_i x_j^{-1} q^n}{1 - t x_i x_j^{-1} q^n}.$$

Then difference operators $H_r^{\mathfrak{gl}_{\ell+1}}$ are self-adjoint with respect to $\langle, \rangle'_{q,t}$:

$$\langle f, H_r^{\mathfrak{gl}_{\ell+1}} g \rangle'_{q,t} = \langle H_r^{\mathfrak{gl}_{\ell+1}} f, g \rangle'_{q,t}.$$

For Macdonald polynomials one has an analog of Cauchy-Littlewood formula

$$C_{\ell+1, m+1}(x, y|q, t) = \sum_{\Lambda} P_\Lambda(x; q, t) P_\Lambda(y; q, t) b_\Lambda(q, t), \quad m \leq \ell,$$

where the sum is over all Young diagrams of \mathfrak{gl}_{m+1} and

$$C_{\ell+1, m+1}(x, y|q, t) = \prod_{i=1}^{\ell+1} \prod_{j=1}^{m+1} \prod_{n=0}^{\infty} \frac{1 - t x_i y_j q^n}{1 - x_i y_j q^n}, \quad (1.2)$$

$$b_\Lambda(q, t) = \frac{1}{\langle P_\Lambda^{\mathfrak{gl}_{\ell+1}}, P_\Lambda^{\mathfrak{gl}_{\ell+1}} \rangle'_{q,t}} = \prod_{n=1}^N \prod_{k=0}^{n-1} \prod_{m=\Lambda_n, n-k}^{\Lambda_n, n-k-1} \frac{1 - q^{m+1} t^k}{1 - q^m t^{k+1}}, \quad \Lambda_{n,j} = \Lambda_n - \Lambda_j, \quad j < n.$$

Proposition 1.1 [AOS] *The following relations hold*

1.

$$\begin{aligned} P_\Lambda^{\mathfrak{gl}_{\ell+1}}(x; q, t) &= \frac{\langle P_\Lambda^{\mathfrak{gl}_\ell}, P_\Lambda^{\mathfrak{gl}_\ell} \rangle'_{q,t}}{\ell! \langle P_\Lambda^{\mathfrak{gl}_\ell}, P_\Lambda^{\mathfrak{gl}_\ell} \rangle'_{q,t}} \\ &\times \oint_{x_1=0} \cdots \oint_{x_{\ell+1}=0} \prod_{i=1}^{\ell} \frac{dy_i}{2\pi i y_i} C_{\ell+1, \ell}(x, y^{-1}|q, t) P_\Lambda^{\mathfrak{gl}_\ell}(y; q, t) \Delta(y|q, t). \end{aligned} \quad (1.3)$$

2.

$$P_{\Lambda}^{\mathfrak{gl}_{\ell+1}}(x; q, t) = \frac{\langle P_{\Lambda}^{\mathfrak{gl}_{\ell+1}}, P_{\Lambda}^{\mathfrak{gl}_{\ell+1}} \rangle_{q,t}}{(\ell+1)! \langle P_{\Lambda}^{\mathfrak{gl}_{\ell+1}}, P_{\Lambda}^{\mathfrak{gl}_{\ell+1}} \rangle'_{q,t}} \times \oint_{x_1=0} \cdots \oint_{x_{\ell+1}=0} \prod_{i=1}^{\ell+1} \frac{dy_i}{2\pi i y_i} C_{\ell+1, \ell+1}(x, y^{-1}|q, t) P_{\Lambda}^{\mathfrak{gl}_{\ell+1}}(y; q, t) \Delta(y|q, t). \quad (1.4)$$

3.

$$P_{\Lambda + (\ell+1)^k}^{\mathfrak{gl}_{\ell+1}}(x; q, t) = \left(\prod_{j=1}^{\ell+1} x_j^k \right) P_{\Lambda}^{\mathfrak{gl}_{\ell+1}}(x; q, t). \quad (1.5)$$

Here $\Lambda + (\ell+1)^k$ is a Young diagram obtained from Λ by a substitution $\Lambda_j \rightarrow \Lambda_j + k$ and

$$\langle P_{\Lambda}^{\mathfrak{gl}_{\ell+1}}, P_{\Lambda}^{\mathfrak{gl}_{\ell+1}} \rangle'_{q,t} = \prod_{1 \leq i < j \leq \ell+1} \prod_{n=0}^{\infty} \frac{1 - t^{j-i} q^{\Lambda_i - \Lambda_j + n}}{1 - t^{j-i+1} q^{\Lambda_i - \Lambda_j + n}} \cdot \frac{1 - t^{j-i} q^{\Lambda_i - \Lambda_j + n+1}}{1 - t^{j-i-1} q^{\Lambda_i - \Lambda_j + 1}},$$

$$\langle P_{\Lambda}^{\mathfrak{gl}_{\ell+1}}, P_{\Lambda}^{\mathfrak{gl}_{\ell+1}} \rangle_{q,t} = \prod_{n=1}^N \prod_{k=0}^{n-1} \prod_{m=\Lambda_{n,n-k}}^{\Lambda_{n,n-k-1}-1} \frac{1 - q^m t^{k+1}}{1 - q^{m+1} t^k}, \quad \Lambda_{n,j} = \Lambda_n - \Lambda_j, \quad j < n.$$

These relations provide a recursive construction of Macdonald polynomials corresponding to arbitrary Young diagrams.

Remark 1.1 Relations (1.3) and (1.4) are analogous to the action of recursion and Baxter operators on Whittaker functions considered in [GLO].

It follows from above considerations that the following intertwining relations hold.

Proposition 1.2 Let $H_k^{\mathfrak{gl}_{\ell+1}}(x)$ and $C_{\ell+1, \ell}(x, y|q, t)$ be given by (1.1) and (1.2) respectively. The following intertwining relations hold

$$H_k^{\mathfrak{gl}_{\ell+1}}(x) C_{\ell+1, \ell}(x, y|q, t) = (t^{k-1} H_{k-1}^{\mathfrak{gl}_{\ell}}(y) + t^k H_k^{\mathfrak{gl}_{\ell}}(y)) C_{\ell+1, \ell}(x, y|q, t), \quad k = 1, \dots, \ell+1. \quad (1.6)$$

2 q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker function

Quantum Hamiltonians of q -deformed $\mathfrak{gl}_{\ell+1}$ -Toda chain can be considered as particular degeneration of Macdonald-Ruijsenaars difference operators [Et]. This is an analog of the Inozemtsev limit [I] producing Toda chains from Calogero-Sutherland models. In this section we give an explicit expression for q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker function obtained using a degeneration of recursive operators for Macdonald polynomials discussed in the previous section. We also consider interesting features of the obtained expressions. Details of the proof will be given in the last section.

Quantum $\mathfrak{gl}_{\ell+1}$ -Toda chain is defined by a set of $\ell+1$ mutually commuting functionally independent quantum Hamiltonians $\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}$, $r = 1, \dots, \ell+1$

$$\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}(x) = \sum_{I_r} (X_{i_1}^{1-\delta_{i_1, 1}} \cdot X_{i_2}^{1-\delta_{i_2, 1}} \cdots X_{i_r}^{1-\delta_{i_r, 1}}) T_{i_1} \cdots T_{i_r}, \quad (2.1)$$

where summation goes over ordered subsets $I_r = \{i_1 < i_2 < \dots < i_r\}$ of $\{1, 2, \dots, \ell + 1\}$ and $X_i(x) := 1 - x_i x_{i-1}^{-1}$, $1 < i \leq \ell + 1$ with $X_1 = 1$, $T_i = q^{x_i \partial_{x_i}}$. The simplest operator is given by

$$\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}(x) = T_1 + \sum_{i=1}^{\ell} (1 - x_{i+1} x_i^{-1}) T_{i+1}.$$

Common eigenfunctions of the Hamiltonians are given by q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker functions [Et]

$$\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}(x) \Psi_{l_1, \dots, l_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(x_1, \dots, x_{\ell+1}) = \left(\sum_{I_r} \prod_{i \in I_r} q^{l_i} \right) \Psi_{l_1, \dots, l_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(x_1, \dots, x_{\ell+1}), \quad (2.2)$$

where l_i , $i = 1, \dots, \ell + 1$ are real numbers.

Note that the set of equations (2.2) allows an infinite-dimensional linear space of solutions (due to a possibility to multiply any solution on an arbitrary function $f(x_1, \dots, x_{\ell+1})$ periodic with respect to the shifts $x_i \rightarrow x_i + m_i$, $m_i \in \mathbb{Z}$). To obtain a finite-dimensional space of solutions we specify the variables to the lattice $\mathbb{Z}^{\ell+1}$ as follows

$$x_j = q^{p_{\ell+1, j} + j - 1}, \quad p_{\ell+1, j} \in \mathbb{Z}, j = 1, \dots, \ell + 1.$$

We shall use the following notation $\underline{p}_{\ell+1} = (p_{\ell+1, 1}, \dots, p_{\ell+1, \ell+1})$. The complete set of commuting Hamiltonians (2.1) can be restricted to the lattice $\mathbb{Z}^{\ell+1}$ using the substitution $X_i(\underline{p}_{\ell+1}) = 1 - q^{p_{\ell+1, i} - p_{\ell+1, i-1} + 1}$, $X_1(\underline{p}_{\ell+1}) = 1$ and $T_i f(\underline{p}_{\ell+1}) = f(\tilde{\underline{p}}_{\ell+1})$ with $\tilde{p}_{\ell+1, k} = p_{\ell+1, k} + \delta_{k, i}$. Thus the first non-trivial Hamiltonian is given by:

$$\mathcal{H}_1^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) = T_1 + \sum_{i=1}^{\ell} (1 - q^{p_{\ell+1, i+1} - p_{\ell+1, i} + 1}) T_{i+1}.$$

We shall be interested in the solution of the eigenvalue problem of q -deformed Toda chain on the lattice $\mathbb{Z}^{\ell+1}$:

$$\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) \Psi_{\underline{l}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) = \left(\sum_{I_r} \prod_{i \in I_r} q^{l_i} \right) \Psi_{\underline{l}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}), \quad (2.3)$$

where $\underline{l} = (l_1, \dots, l_{\ell+1})$.

Let $\mathcal{P}^{(\ell+1)}$ be a set of collections of integers $p_{i, j} \in \mathbb{Z}$, $i = 1, \dots, \ell + 1$, $j = 1, \dots, i$ satisfying the conditions $p_{i+1, j} \leq p_{i, j} \leq p_{i+1, j+1}$ with fixed $p_{\ell+1, i}$, $i = 1, \dots, \ell + 1$. Thus $\mathcal{P}^{(\ell+1)}$ is a set of Gelfand-Zetlin patterns corresponding to an irreducible finite-dimensional representation of $GL(\ell + 1, \mathbb{C})$ (see e.g. [ZS]). We denote by $\mathcal{P}_{\ell+1, \ell} \subset \mathcal{P}^{(\ell+1)}$ the subset $p_{\ell+1, i} \leq p_{\ell, i} \leq p_{\ell+1, i+1}$, $i = 1, \dots, \ell$ of $\mathcal{P}^{(\ell+1)}$.

Theorem 2.1 *The following function is a solution of the eigenfunction problem (2.3)*

$$\begin{aligned} \Psi_{\underline{l}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) &= \sum_{p_{k, i} \in \mathcal{P}^{(\ell+1)}} \prod_{k=1}^{\ell+1} q^{l_k (\sum_{i=1}^k p_{k, i} - \sum_{i=1}^{k-1} p_{k-1, i})} \\ &\times \frac{\prod_{k=2}^{\ell} \prod_{i=1}^{k-1} (p_{k, i+1} - p_{k, i})_q!}{\prod_{k=1}^{\ell} \prod_{i=1}^k (p_{k, i} - p_{k+1, i})_q! (p_{k+1, i+1} - p_{k, i})_q!}, \quad p_{\ell+1, 1} \leq \dots \leq p_{\ell+1, \ell+1}, \\ \Psi_{\underline{l}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) &= 0, \quad \text{otherwise.} \end{aligned} \quad (2.4)$$

Here we set $(n)_q! = (1-q)\dots(1-q^n)$.

The proof of the theorem will be given in Section 4.

Example 2.1 Let $\mathfrak{g} = \mathfrak{gl}_2$ and $(p_{2,1}, p_{2,2}) \in \mathbb{Z}^2$. The function

$$\Psi_{l_1, l_2}^{\mathfrak{gl}_2}(p_{2,1}, p_{2,2}) = \sum_{p_{2,1} \leq p_{1,1} \leq p_{2,2}} \frac{q^{l_1 p_{1,1}} q^{l_2(p_{2,1} + p_{2,2} - p_{1,1})}}{(p_{1,1} - p_{2,1})_q! (p_{2,2} - p_{1,1})_q!}, \quad p_{2,1} \leq p_{2,2},$$

$$\Psi_{l_1, l_2}^{\mathfrak{gl}_2}(p_{2,1}, p_{2,2}) = 0, \quad p_{2,1} > p_{2,2},$$

is a common eigenfunction of commuting Hamiltonians

$$\mathcal{H}_1^{\mathfrak{gl}_2} = T_1 + (1 - q^{p_{2,2} - p_{2,1} + 1})T_2, \quad \mathcal{H}_2^{\mathfrak{gl}_2} = T_1 T_2.$$

Note that the formula (2.4) can be easily rewritten in the recursive form.

Corollary 2.1 The following recursive relation holds

$$\Psi_{\underline{l}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) = \sum_{p_{\ell,i} \in \mathcal{P}_{\ell+1,\ell}} \Delta(\underline{p}_{\ell}) q^{l_{\ell+1}(\sum_{i=1}^{\ell+1} p_{\ell+1,i} - \sum_{i=1}^{\ell} p_{\ell,i})} \mathcal{Q}_{\ell+1,\ell}(\underline{p}_{\ell+1}, \underline{p}_{\ell} | q) \Psi_{\underline{l}'}^{\mathfrak{gl}_{\ell}}(\underline{p}_{\ell}), \quad (2.5)$$

where

$$\mathcal{Q}_{\ell+1,\ell}(\underline{p}_{\ell+1}, \underline{p}_{\ell} | q) = \frac{1}{\prod_{i=1}^{\ell} (p_{\ell,i} - p_{\ell+1,i})_q! (p_{\ell+1,i+1} - p_{\ell,i})_q!},$$

and

$$\Delta(\underline{p}_{\ell}) = \prod_{i=1}^{\ell-1} (p_{\ell,i+1} - p_{\ell,i})_q!,$$

where the notations $\underline{l} = (l_1, \dots, l_{\ell+1})$, $\underline{l}' = (l_1, \dots, l_{\ell})$ are used.

Remark 2.1 In the limit $q \rightarrow 1$ the q -deformed Toda chain eigenfunction equations reduce to ordinary Toda chain eigenfunction equations. The solution (2.4) in the limit $q \rightarrow 1$ reduces to an integral representation of $\mathfrak{gl}_{\ell+1}$ -Whittaker functions due to Givental [Gi]

$$\Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(x_1, \dots, x_{\ell+1}) = \int_{\mathbb{R}^{\frac{\ell(\ell+1)}{2}}} \prod_{k=1}^{\ell} \prod_{i=1}^k dx_{k,i} e^{\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x)}, \quad (2.6)$$

where

$$\mathcal{F}^{\mathfrak{gl}_{\ell+1}}(x) = \sum_{k=1}^{\ell+1} \lambda_k \left(\sum_{i=1}^k x_{k,i} - \sum_{i=1}^{k-1} x_{k-1,i} \right) - \sum_{k=1}^{\ell} \sum_{i=1}^k \left(e^{x_{k,i} - x_{k+1,i}} + e^{x_{k+1,i+1} - x_{k,i}} \right),$$

$\underline{l} = (l_1, \dots, l_{\ell+1})$ and $x_i := x_{\ell+1,i}$, $i = 1, \dots, \ell+1$.

Proposition 2.1 *The common eigenfunction (2.4) of q -deformed Toda chain allows the following representation for $p_{\ell+1,1} \leq p_{\ell+1,2} \leq \dots p_{\ell+1,\ell+1}$*

$$\Psi_{\underline{l}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) = \text{Tr}_V q^d \prod_{i=1}^{\ell+1} q^{l_i E_{i,i}}, \quad (2.7)$$

where V is a $\mathbb{C}^* \times GL(\ell+1, \mathbb{C})$ -module, $E_{i,i}$, $i = 1, \dots, \ell+1$ are Cartan generators of $\mathfrak{gl}_{\ell+1} = \text{Lie}(GL(\ell+1, \mathbb{C}))$ and d is a generator of $\text{Lie}(\mathbb{C}^*)$.

Proof. It is useful to rewrite (2.4) in the following form

$$\begin{aligned} \Psi_{\underline{l}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) &= \Delta(\underline{p}_{\ell+1})^{-1} \sum_{p_{k,i} \in \mathcal{P}^{(\ell+1)}} \prod_{k=1}^{\ell} q^{l_{k+1}(\sum_i p_{k+1,i} - \sum_j p_{k,i})} \prod_{i=1}^k \binom{p_{k+1,i+1} - p_{k+1,i}}{p_{k,i} - p_{k+1,i}}_q, \\ \Delta(\underline{p}_{\ell+1}) &= \prod_{j=1}^{\ell} (p_{\ell+1,j+1} - p_{\ell+1,j})_{q!}, \end{aligned}$$

where

$$\binom{n}{k}_q = \frac{(n)_q!}{(n-k)_q! (k)_q!}.$$

Now taking into account the identities

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q, \quad \frac{1}{(1-q^n)} = \sum_{k=0}^{\infty} q^{kn},$$

one obtains a representation of $\Psi_{\underline{l}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1})$ as a sum of terms $q^N z_1^{m_1} \dots z_{\ell+1}^{m_{\ell+1}}$ with positive integer coefficients. Let us note that q -deformed Toda chain eigenfunction equations depend on variables $z_i = q^{l_i}$ only through characters of fundamental representations of $\mathfrak{gl}_{\ell+1}$. Checking that the initial conditions leading to the solution (2.4) can be also expressed through characters and using the expansion with positive integral coefficients discussed above, one obtains the representation (2.7) \square

Remark 2.2 *The following representation holds for $p_{\ell+1,1} \leq p_{\ell+1,2} \leq \dots p_{\ell+1,\ell+1}$*

$$\tilde{\Psi}_{\underline{l}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) = \Delta(\underline{p}_{\ell+1}) \Psi_{\underline{l}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) = \text{Tr}_{V_f} q^d \prod_{i=1}^{\ell+1} q^{l_i E_{i,i}}, \quad (2.8)$$

where V_f is a finite-dimensional $\mathbb{C}^* \times GL(\ell+1, \mathbb{C})$ -module (see Proposition 3.4 for explicit description of V_f). The module V entering (2.7) and the module V_f entering (2.8) have a more refined structure under the action of (quantum) affine Lie algebras and will be discussed in an other part of this paper [GLO2].

3 Non-standard limits

Besides the limit $q \rightarrow 1$ recovering classical $\mathfrak{gl}_{\ell+1}$ -Whittaker function as a solution of $\mathfrak{gl}_{\ell+1}$ -Toda chain there are other interesting limits elucidating the meaning of q -deformed Toda chain equations. In the limit $q \rightarrow 0$ the q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker functions are given by characters of irreducible representations of $\mathfrak{gl}_{\ell+1}$. This allows to identify the Whittaker functions with p -adic Whittaker functions according to Shintani-Casselman-Shalika formula [Sh], [CS]. There is also a non-standard $q \rightarrow 1$ limit which clarifies the recursive structure of q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker functions.

3.1 The limit $q \rightarrow 0$

In this subsection we discuss a limit $q \rightarrow 0$ of the constructed q -deformed Whittaker functions (we restrict Whittaker function to the domain $\{p_{\ell+1,1} \leq \dots \leq p_{\ell+1,\ell+1}\}$ where it is non-trivial). We will show that in the domain $\{p_{\ell+1,1} \leq \dots \leq p_{\ell+1,\ell+1}\}$ the system of equations for a common eigenfunctions of q -deformed Toda chain Hamiltonians reduces to Pieri formulas (particular case of Littlewood-Richardson rules) for the decomposition of the tensor product of an arbitrary finite-dimensional representation and a fundamental representation of $\mathfrak{gl}_{\ell+1}$.

Proposition 3.1 *1. In the limit $q \rightarrow 0$, the solution (2.4) is given in the domain $p_{\ell+1,1} \leq \dots \leq p_{\ell+1,\ell+1}$ by*

$$\Psi_{\underline{l}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1})|_{q \rightarrow 0} := \chi_{\underline{p}_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(\underline{z}) = \sum_{p_{k,i} \in \mathcal{P}^{\ell+1}} \prod_{k=1}^{\ell+1} z_k^{(\sum_{i=1}^k p_{k,i} - \sum_{i=1}^{k-1} p_{k-1,i})}, \quad (3.1)$$

where we set $z_i = q^{l^i}$, $i = 1, \dots, \ell + 1$.

2. Functions $\chi_{\underline{p}_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(z)$ satisfy the following set of difference equations

$$\chi_r^{\mathfrak{gl}_{\ell+1}}(\underline{z}) \chi_{\underline{p}_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(\underline{z}) = \sum_{I_r} \chi_{\underline{p}_{\ell+1} + I_r}^{\mathfrak{gl}_{\ell+1}}(\underline{z}), \quad (3.2)$$

where $\underline{z} = (z_1, z_2, \dots, z_{\ell+1})$.

3. The functions $\chi_{\underline{p}_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(\underline{z})$ can be identified with characters of irreducible finite-dimensional representations of $GL_{\ell+1}$ corresponding to partitions $p_{\ell+1,1} \leq \dots \leq p_{\ell+1,\ell+1}$.

Proof: The relations (3.1) and (3.2) follows directly from the similar relations for generic q . To prove the last statement note that (3.1) can be identified with expression for characters of irreducible finite-dimensional representations of $GL_{\ell+1}$ obtained using the Gelfand-Zetlin bases (see e.g. [ZS]). Let $\{p_{ij}\}$, $i = 1, \dots, \ell + 1$, $j = 1, \dots, i$ be a Gelfand-Zetlin (GZ) pattern $\mathcal{P}^{(\ell+1)}$ i.e. satisfy the conditions $p_{i+1,j} \leq p_{i,j} \leq p_{i+1,j+1}$. Irreducible finite-dimensional representation can be realized in a vector space with the bases $v_{\underline{p}}$ parametrized by GZ patterns $\{p_{ij}\}$ with fixed $p_{\ell+1,i}$. Action of Cartan generators on $v_{\underline{p}}$ is given by

$$z_1^{E_{11}} z_2^{E_{22}} \dots z_{\ell+1}^{E_{\ell+1,\ell+1}} v_{\underline{p}} = z_1^{s_1} z_2^{s_2 - s_1} \dots z_{\ell+1}^{s_{\ell+1} - s_{\ell}} v_{\underline{p}}, \quad s_k = \sum_{i=1}^k p_{ki}. \quad (3.3)$$

Thus we have for the character

$$\chi_{p_{\ell+1,1}, \dots, p_{\ell+1,\ell+1}}^{\mathfrak{gl}_{\ell+1}}(z_1, \dots, z_{\ell+1}) = \sum_{p_{k,i} \in \mathcal{P}^{(\ell+1)}} \prod_{k=1}^{\ell+1} z_k^{(\sum_{i=1}^k p_{k,i} - \sum_{i=1}^{k-1} p_{k-1,i})} \quad (3.4)$$

□

Remark 3.1 *The second identity in Proposition 3.1 is known as the Pieri formula (see e.g. [FH], Appendix A). Thus the q -deformed Toda chain equations can be considered as q -deformations of Pieri's formula. There is a generalization of q -Toda chain relations providing a q -version of a general Littlewood-Richardson rule.*

The expressions for the characters in GZ representation have an obvious recursive structure which is a $q \rightarrow 0$ limit of (2.5).

Corollary 3.1 *Characters satisfy the following recursive relation*

$$\chi_{p_{\ell+1,1}, \dots, p_{\ell+1, \ell+1}}^{\mathfrak{gl}_{\ell+1}}(z_1, \dots, z_{\ell+1}) = \sum_{p_{\ell, i} \in \mathcal{P}_{\ell+1, \ell}} z_{\ell+1}^{\sum_{i=1}^{\ell+1} p_{\ell+1, i} - \sum_{i=1}^{\ell} p_{\ell, i}} \chi_{p_{\ell, 1}, \dots, p_{\ell, \ell}}^{\mathfrak{gl}_{\ell}}(z_1, \dots, z_{\ell}), \quad (3.5)$$

where sum goes over $\underline{p}_{\ell} = (p_{\ell, 1}, \dots, p_{\ell, \ell})$ satisfying the GZ conditions:

$$p_{\ell+1, i} \leq p_{\ell, i} \leq p_{\ell+1, i+1}.$$

Note that these recursive relations can be derived using the classical Cauchy-Littlewood formula

$$C_{\ell+1, m+1}(x, y) = \prod_{i=1}^{\ell+1} \prod_{j=1}^{m+1} \frac{1}{1 - x_i y_j} = \sum_{\Lambda} \chi_{\Lambda}^{\mathfrak{gl}_{\ell+1}}(x) \chi_{\Lambda}^{\mathfrak{gl}_{m+1}}(y), \quad (3.6)$$

where the sum goes over Young diagrams Λ of \mathfrak{gl}_{m+1} and $\chi_{\Lambda}^{\mathfrak{gl}_{\ell+1}}(x) = \chi_{\Lambda}^{\mathfrak{gl}_{\ell+1}}(x_1, \dots, x_{\ell+1})$ are characters of irreducible finite-dimensional representation of $GL(\ell+1, \mathbb{C})$ corresponding to Young diagrams Λ .

Proposition 3.2 *The following integral relations for the characters $\chi_{\Lambda}^{\mathfrak{gl}_{\ell+1}}(x)$ hold*

$$\begin{aligned} \chi_{\Lambda}^{\mathfrak{gl}_{\ell+1}}(x) &= \oint_{y_1=0} \cdots \oint_{y_{\ell}=0} \prod_{i=1}^{\ell} \frac{dy_i}{2\pi i y_i} C_{\ell+1, \ell}(x, y^{-1}) \chi_{\Lambda}^{\mathfrak{gl}_{\ell}}(y) \Delta(y), \\ \chi_{\Lambda}^{\mathfrak{gl}_{\ell+1}}(x) &= \oint_{y_1=0} \cdots \oint_{y_{\ell+1}=0} \prod_{i=1}^{\ell+1} \frac{dy_i}{2\pi i y_i} C_{\ell+1, \ell+1}(x, y^{-1}) \chi_{\Lambda}^{\mathfrak{gl}_{\ell+1}}(y) \Delta(y), \\ \chi_{\Lambda + (\ell+1)^k}^{\mathfrak{gl}_{\ell+1}}(x) &= \left(\prod_{j=1}^{\ell+1} x_j^k \right) \chi_{\Lambda}^{\mathfrak{gl}_{\ell+1}}(x). \end{aligned} \quad (3.7)$$

The relations above allow to obtain character of irreducible finite-dimensional representation of $GL(\ell+1, \mathbb{C})$ corresponding to any Young diagram Λ .

Remark 3.2 *The relations above can be obtained from similar relations for Macdonald polynomials in the limit $t \rightarrow 0$, $q \rightarrow 0$. These recursion relations are analogs of Mellin-Barnes recursion relations for classical Whittaker functions (see [KL1], [GKL], [GLO] for details).*

According to Shintani-Casselman-Shalika formula, the p -adic Whittaker function corresponding to a Lie group G is equal to the character of the Langlands dual Lie group ${}^L G$ acting in an irreducible finite-dimensional representation [Sh], [CS]. Thus according to Proposition 3.1 we can consider $\mathfrak{gl}_{\ell+1}$ -Whittaker functions at $q \rightarrow 0$ as an incarnation of p -adic Whittaker functions (this is in complete agreement with the results of [GLO3]. Moreover, taking into account Proposition 2.1 one can consider the main result of this paper as a generalization of Shintani-Casselman-Shalika formula to a q -deformed case including a limiting case of classical $\mathfrak{gl}_{\ell+1}$ -Whittaker functions. This interpretation of classical Whittaker functions evidently deserves further attention.

3.2 Modified limit $q \rightarrow 1$

In this subsection we consider a modified limit $q \rightarrow 1$ leading to a very simple degeneration of q -deformed Toda chain. In this limit q -deformed Toda chain can be easily solved. Moreover the form of the solution makes the recursive expressions (2.5) for q -deformed Toda chain solution very natural.

Let us redefine the q -deformed Toda chain Hamiltonians and their common eigenfunctions as follows (we assume $p_{\ell+1,1} \leq \dots \leq p_{\ell+1,\ell+1}$ below)

$$\mathcal{J}_r^{\mathfrak{gl}_{\ell+1}} = \Delta(\underline{p}_{\ell+1}) \mathcal{H}_r^{\mathfrak{gl}_{\ell+1}} \Delta(\underline{p}_{\ell+1})^{-1},$$

$$\tilde{\Psi}_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) = \Delta(\underline{p}_{\ell+1}) \cdot \Psi_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}), \quad \Delta(\underline{p}_{\ell+1}) = \prod_{j=1}^{\ell} (p_{\ell+1,j+1} - p_{\ell+1,j})_q!.$$

Explicitly we have

$$\mathcal{J}_r^{\mathfrak{gl}_{\ell+1}} = \sum_{I_r} (\tilde{X}_{i_1}^{1-\delta_{i_2-i_1,1}} \dots \tilde{X}_{i_{r-1}}^{1-\delta_{i_r-i_{r-1},1}} \cdot \tilde{X}_{i_r}^{1-\delta_{i_{r+1}-i_r,1}}) T_{i_1} \dots T_{i_r}, \quad (3.8)$$

and we assume $i_{r+1} = \ell + 2$ and $\tilde{X}_i = 1 - q^{p_{i+1}-p_i}$. Let us now take the limit $q \rightarrow 1$

$$\tilde{\psi}_{\underline{\lambda}}^{(\ell+1)}(\underline{p}_{\ell+1}) = \lim_{q \rightarrow 1} \tilde{\Psi}_{\underline{\lambda}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}), \quad h_r^{(\ell+1)} = \lim_{q \rightarrow 1} \mathcal{J}_r^{\mathfrak{gl}_{\ell+1}}.$$

We have $\lim_{q \rightarrow 1} (1 - q^n) = 0$, and therefore we obtain from (2.1)

$$h_r^{(\ell+1)} = T_{\ell+2-r} \dots T_{\ell+1}.$$

Now the eigenfunction problem is easily solved.

Proposition 3.3 1. *The function*

$$\tilde{\psi}_{\underline{\lambda}}^{(\ell+1)}(\underline{p}_{\ell+1}) = \left(\chi_{\ell+1}^{\mathfrak{gl}_{\ell+1}}(\underline{z}) \right)^{p_{\ell+1,1}} \prod_{i=1}^{\ell} \left(\chi_{\ell+1-i}^{\mathfrak{gl}_{\ell+1}}(\underline{z}) \right)^{p_{\ell+1,i+1} - p_{\ell+1,i}} \quad (3.9)$$

is an eigenfunction of the family of mutually commuting difference operators

$$h_r^{(\ell+1)} \tilde{\psi}_{\underline{\lambda}}^{(\ell+1)}(\underline{p}_{\ell+1}) = \chi_r^{\mathfrak{gl}_{\ell+1}}(\underline{z}) \tilde{\psi}_{\underline{\lambda}}^{(\ell+1)}(\underline{p}_{\ell+1}), \quad (3.10)$$

where $\chi_r^{\mathfrak{gl}_{\ell+1}}(\underline{z})$ is the character of fundamental representation $V_{\omega_r} = \bigwedge^r \mathbb{C}^{\ell+1}$:

$$\chi_r^{\mathfrak{gl}_{\ell+1}}(\underline{z}) = \sum_{I_r} z_{i_1} \dots z_{i_r}, \quad z_i = q^{\lambda_i}, \quad i = 1, \dots, \ell+1,$$

and

$$h_r^{(\ell+1)} = T_{\ell+2-r} \dots T_{\ell+1}, \quad r = 1, \dots, \ell+1.$$

2. *In the domain $p_{\ell+1,1} \leq \dots \leq p_{\ell+1,\ell+1}$ the following recursive relation holds*

$$\tilde{\psi}_{\underline{\lambda}}^{(\ell+1)}(\underline{p}_{\ell+1}) = \sum_{p_{\ell,i} \in \mathcal{P}_{\ell+1,\ell}} z_{\ell+1}^{\sum_{i=1}^{\ell+1} p_{\ell+1,i} - \sum_{i=1}^{\ell} p_{\ell,i}} \prod_{i=1}^{\ell} \binom{p_{\ell+1,i+1} - p_{\ell+1,i}}{p_{\ell,i} - p_{\ell+1,i}} \cdot \tilde{\psi}_{\underline{\lambda}'}^{(\ell)}(\underline{p}_{\ell}), \quad (3.11)$$

where $\underline{\lambda}' = (\lambda_1, \dots, \lambda_{\ell})$ and $z_{\ell+1} = q^{\lambda_{\ell+1}}$.

Proof: The identity (3.10) follows from the construction. Let us prove that (3.11) follows from (3.10). Denote $(z_1, \dots, z_{\ell+1}) = (q^{l_1}, \dots, q^{l_{\ell+1}})$. Using the relation

$$\chi_r^{\mathfrak{gl}_{\ell+1}}(\underline{z}) = \chi_r^{\mathfrak{gl}_{\ell}}(\underline{z}') + z_{\ell+1} \cdot \chi_{r-1}^{\mathfrak{gl}_{\ell}}(\underline{z}'), \quad r = 1, \dots, \ell + 1,$$

we have

$$\begin{aligned} & \left(\chi_{\ell+1}^{\mathfrak{gl}_{\ell+1}}(\underline{z}) \right)^{p_{\ell+1,1}} \prod_{i=1}^{\ell} \left(\chi_{\ell+1-i}^{\mathfrak{gl}_{\ell+1}}(\underline{z}) \right)^{p_{\ell+1,i+1}-p_{\ell+1,i}} \\ &= \left(z_{\ell+1} \chi_{\ell}^{\mathfrak{gl}_{\ell}}(\underline{z}') \right)^{p_{\ell+1,1}} \prod_{i=1}^{\ell} \sum_{p_{\ell,i}=p_{\ell+1,i}}^{p_{\ell+1,i+1}} \left(z_{\ell+1} \chi_{\ell-i}^{\mathfrak{gl}_{\ell}}(\underline{z}') \right)^{p_{\ell+1,i+1}-p_{\ell+1,i}} \\ & \quad \cdot \left(\chi_{\ell+1-i}^{\mathfrak{gl}_{\ell}}(\underline{z}') \right)^{p_{\ell,i}-p_{\ell+1,i}} \cdot \binom{p_{\ell+1,i+1} - p_{\ell+1,i}}{p_{\ell,i} - p_{\ell+1,i}} \\ &= \sum_{p_{\ell,i} \in \mathcal{P}_{\ell+1,\ell}} z_{\ell+1}^{\sum_i p_{\ell+1,i+1} - \sum_i p_{\ell,i}} \prod_{i=1}^{\ell} \binom{p_{\ell+1,i+1} - p_{\ell+1,i}}{p_{\ell,i} - p_{\ell+1,i}} \\ & \quad \cdot \left(\chi_{\ell}^{\mathfrak{gl}_{\ell}}(\underline{z}') \right)^{p_{\ell,1}} \prod_{i=1}^{\ell-1} \left(\chi_{\ell-i}^{\mathfrak{gl}_{\ell}}(\underline{z}') \right)^{p_{\ell,i+1}-p_{\ell,i}} \\ &= \sum_{p_{\ell,i} \in \mathcal{P}_{\ell+1,\ell}} z_{\ell+1}^{\sum_i p_{\ell+1,i+1} - \sum_i p_{\ell,i}} \prod_{i=1}^{\ell} \binom{p_{\ell+1,i+1} - p_{\ell+1,i}}{p_{\ell,i} - p_{\ell+1,i}} \tilde{\psi}_{\underline{\lambda}'}^{(\ell)}(\underline{p}_{\ell}). \end{aligned} \tag{3.12}$$

Thus we obtain the recursion formula described in the proposition \square

Remark 3.3 *The functions*

$$\psi_{\underline{\lambda}}^{(\ell+1)}(\underline{p}_{\ell+1}) = \Delta^{-1}(\underline{p}_{\ell+1}) \tilde{\psi}_{\underline{\lambda}}^{(\ell+1)}(\underline{p}_{\ell+1}),$$

satisfy following recursive relations

$$\psi_{\underline{\lambda}}^{(\ell+1)}(\underline{p}_{\ell+1}) = \sum_{p_{\ell,i} \in \mathcal{P}_{\ell+1,\ell}} z_{\ell+1}^{\sum_i p_{\ell+1,i} - \sum_i p_{\ell,i}} \prod_{i=1}^{\ell-1} \frac{(p_{\ell,i+1} - p_{\ell,i})!}{(p_{\ell,i} - p_{\ell+1,i})! (p_{\ell+1,i+1} - p_{\ell,i})!} \psi_{\underline{\lambda}'}^{(\ell)}(\underline{p}_{\ell}). \tag{3.13}$$

This makes the formula (2.5) for the solution of q -deformed Toda chain slightly less mysterious.

Proposition 3.4 *The following representation holds*

$$\tilde{\psi}_{\underline{\lambda}}^{(\ell+1)}(\underline{p}_{\ell+1}) = \text{Tr}_{V_f} \prod_{i=1}^{\ell+1} q^{l_i E_{i,i}}, \tag{3.14}$$

where

$$V_f = V_{\omega_1}^{\otimes(p_{\ell+1,\ell+1}-p_{\ell+1,\ell})} \otimes \dots \otimes V_{\omega_{\ell}}^{\otimes(p_{\ell+1,2}-p_{\ell+1,1})} \otimes V_{\omega_{\ell+1}}^{\otimes p_{\ell+1,1}}, \tag{3.15}$$

where $V_{\omega_n} = \wedge^n \mathbb{C}$ are the fundamental representations of $GL(\ell+1, \mathbb{C})$.

Proof: Obvious consequence of the Proposition 3.3 \square

The module V_f entering (2.8) is isomorphic to (3.15) as $GL(\ell+1, \mathbb{C})$ -module but has a more refined structure under the action of quantum affine Lie algebras and will be discussed in [GLO2].

4 Proof of Theorem 2.1

In this section we provide a proof of Theorem 2.1. To derive explicit expression (2.4) for q -deformed $\mathfrak{gl}_{\ell+1}$ -Whittaker function we take a limit $t \rightarrow \infty$ of recursive relations (1.3) for Macdonald polynomials. We start with some useful relations that will be used in the proof of Theorem 2.1.

4.1 q -deformed Toda chain from Macdonald-Ruijsenaars system

In this subsection we demonstrate that quantum Hamiltonians of q -deformed Toda chain arise as a limit of Macdonald-Ruijsenaars operators when $t \rightarrow \infty$. Let us take $t = q^{-k}$.

Proposition 4.1 *The following relations hold*

$$\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}} = \lim_{k \rightarrow \infty} H_{r,k}^{\mathfrak{gl}_{\ell+1}} = \sum_{I_r} (X_{i_1}^{1-\delta_{i_1,1}} \cdot X_{i_2}^{1-\delta_{i_2-i_1,1}} \cdot \dots \cdot X_{i_r}^{1-\delta_{i_r-i_{r-1},1}}) T_{i_1} \cdot \dots \cdot T_{i_r},$$

where

$$H_{r,k}^{\mathfrak{gl}_{\ell+1}} = D(x)^{-1} H_r^{\mathfrak{gl}_{\ell+1}}(x_i q^{ki}) D(x), \quad D(x) = \prod_{i=1}^{\ell+1} x_i^{-k(\ell+1-i)},$$

and the sum is over subsets $I_r = \{i_1 < i_2 < \dots < i_r\} \subset \{1, 2, \dots, \ell+1\}$. We take $X_i = 1 - x_i x_{i-1}^{-1}$, $i = 2, \dots, \ell+1$, with $X_1 = 1$ and $T_i x_j = q^{\delta_{i,j}} x_j T_i$.

Proof: Make a change of variables x_i : $x_i \mapsto x_i t^{-i}$, $i = 1, \dots, \ell+1$. Then for any i and any I_r , containing i we have:

$$\left(\prod_{j \notin I_r} \frac{tx_i - x_j}{x_i - x_j} \right) \mapsto \left(t^{b_{r,i}} \prod_{j > i} \frac{x_i - x_j t^{i-1-j}}{x_i - x_j t^{i-j}} \times \frac{x_i - x_{i-1}}{x_i t^{-1} - x_{i-1}} \times \prod_{j < i-1} \frac{x_i t^{j+1-i} - x_j}{x_i t^{j-i} - x_j} \right), \quad (4.1)$$

where $b_{r,i} = |\{j \notin I_r | j > i\}|$. Making a substitution $t = q^{-k}$ and conjugating the Hamiltonians $H_r^{\mathfrak{gl}_{\ell+1}}$ by

$$D(x) = \prod_{i=1}^{\ell+1} x_i^{-k\varrho_i},$$

leads to a multiplication of each term (4.1) in the sum (1.1) by $\prod_{i \in I_r} q^{-k\varrho_i}$, $\varrho_i := \ell+1-i$. Taking into account that for any i and for any subset I_r containing i one has $\sum_{i \in I_r} (\varrho_i - b_{r,i}) = r(r-1)/2$ we obtain in the limit $k \rightarrow \infty$

$$\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}} = \sum_{I_r} (X_{i_1}^{1-\delta_{i_1,1}} \cdot X_{i_2}^{1-\delta_{i_2-i_1,1}} \cdot \dots \cdot X_{i_r}^{1-\delta_{i_r-i_{r-1},1}}) T_{i_1} \cdot \dots \cdot T_{i_r},$$

where $X_i = 1 - x_i x_{i-1}^{-1}$, $i = 2, \dots, \ell+1$ and $X_1 = 1$ \square

4.2 The Cauchy-Littlewood kernel $C_{\ell+1,\ell}(x, y|q, t)$ for q -deformed Whittaker functions

In this subsection by taking an appropriate limit of the Cauchy-Littlewood kernel for Macdonald polynomials we derive its analog for q -deformed Whittaker functions and verify the intertwining relations with q -deformed Toda chain Hamiltonians.

Let $t = q^{-k}$, $\varrho_i = \ell + 1 - i$. Given the Cauchy-Littlewood kernel $C_{\ell+1,\ell}(x, y|q, t)$ (1.2), define new kernel as

$$Q_{\ell+1,\ell}(x, y|q) = \lim_{k \rightarrow \infty} \left\{ \prod_{i=1}^{\ell} (x_i y_{\ell+1-i})^{-k \varrho_i} \cdot R_k(q) \cdot C_{\ell+1,\ell}(x, y|q, q^{-k}) \right\}, \quad (4.2)$$

where

$$R_k(q) = \prod_{j=1}^k \frac{-q^{a_j}}{(1 - q^j)^{2\ell}}, \quad a_j = \frac{\ell(\ell+1)}{2} \left(j + \frac{\ell-1}{3} k \right).$$

Proposition 4.2 *Then the following explicit expression for $Q_{\ell+1,\ell}(x, y|q)$ defined by (4.2) holds:*

$$Q_{\ell+1,\ell}(x, y|q) = \prod_{i=1}^{\ell} \prod_{n=1}^{\infty} \frac{1 - (x_i y_i)^{-1} q^n}{1 - q^n} \cdot \prod_{i=1}^{\ell} \prod_{n=1}^{\infty} \frac{1 - x_{i+1} y_i q^{-1} q^n}{1 - q^n}. \quad (4.3)$$

Proof: Making substitution $x_i \rightarrow x_i t^{-i}$, $y_i \rightarrow y_i t^i$ in $C_{\ell+1,\ell}(x, y|q, t)$ and taking $t = q^{-k}$ we have

$$\begin{aligned} C_{\ell+1,\ell}(x, y|q, t) &= \prod_{n=0}^{\infty} \prod_{i=1}^{\ell} \frac{1 - x_i y_i q^{n-k}}{1 - x_i y_i q^n} \frac{1 - x_{i+1} y_i q^n}{1 - x_{i+1} y_i q^{n+k}} \\ &\times \prod_{i=3}^{\ell+1} \prod_{j=1}^{i-2} \frac{1 - x_i y_j q^{n+(i-j-1)k}}{1 - x_i y_j q^{n+(i-j)k}} \frac{1 - x_{\ell+2-i} y_{\ell+1-j} q^{n+(j-i)k}}{1 - x_{\ell+2-i} y_{\ell+1-j} q^{n+(j+1-i)k}}. \end{aligned}$$

One encounters four types of factors which can be rewritten as

$$\begin{aligned} \prod_{n=0}^{\infty} \frac{1 - xyq^{n-k}}{1 - xyq^n} &= \prod_{j=1}^k (1 - xyq^{-j}) = (xy)^k \prod_{j=1}^k (-q^{-j})(1 - (xy)^{-1}q^j), \\ \prod_{n=0}^{\infty} \frac{1 - xyq^n}{1 - xyq^{n+k}} &= \prod_{n=0}^{k-1} (1 - xyq^n) = \prod_{j=1}^k (1 - xyq^{-1}q^j), \\ \prod_{n=0}^{\infty} \frac{1 - xyq^{n-(m+1)k}}{1 - xyq^{n-mk}} &= \prod_{j=k+1}^{2k} (1 - xyq^{-j}) = (xy)^k \prod_{j=k+1}^{2k} (-q^{-j})(1 - (xy)^{-1}q^j), \\ \prod_{n=0}^{\infty} \frac{1 - xyq^{n+mk}}{1 - xyq^{n+(m+1)k}} &= \prod_{j=k+1}^{2k} (1 - xyq^{-1}q^j). \end{aligned}$$

Now it is easy to take the limit $k \rightarrow \infty$ and obtain (4.3) \square

Let us introduce a set of slightly modified mutually commuting Hamiltonians

$$\tilde{\mathcal{H}}_r^{\text{gl}_{\ell}}(y) = \sum_{I_r} (Y_{i_1}^{1-\delta_{i_2-i_1,1}} \cdots Y_{i_{r-1}}^{1-\delta_{i_r-i_{r-1},1}} \cdot Y_{i_r}^{1-\delta_{i_{r+1}-i_r,1}}) T_{i_1} \cdots T_{i_r}, \quad (4.4)$$

where

$$Y_i(y) = 1 - y_i y_{i+1}^{-1}, \quad 1 \leq i < \ell, \quad Y_{\ell} = 1.$$

We assume here $I_r = (i_1 < i_2 < \dots < i_r) \subset \{1, 2, \dots, \ell\}$ and we set $i_{r+1} = \ell + 1$.

Proposition 4.3 *The following intertwining relations hold*

$$\mathcal{H}_k^{\mathfrak{gl}_{\ell+1}}(x) Q_{\ell+1,\ell}(x, y|q) = \left\{ \tilde{\mathcal{H}}_{k-1}^{\mathfrak{gl}_{\ell}}(y) + \tilde{\mathcal{H}}_k^{\mathfrak{gl}_{\ell}}(y) \right\} Q_{\ell+1,\ell}(x, y|q), \quad (4.5)$$

where $k = 1, \dots, \ell + 1$. Here $Q_{\ell+1,\ell}(x, y|q)$, $\mathcal{H}_k^{\mathfrak{gl}_{\ell+1}}(x)$ and $\tilde{\mathcal{H}}_k^{\mathfrak{gl}_{\ell}}(y)$ are defined by (4.3), (2.1) and (4.4) respectively.

Proof: Direct calculation similar to the one used in the proof of Proposition 4.1 \square

Let us introduce a function $\mathcal{Q}_{\ell+1,\ell}(\underline{p}_{\ell+1}, \underline{p}_{\ell}|q)$ on the lattice $\mathbb{Z}^{\ell+1} \times \mathbb{Z}^{\ell}$ as follows

$$\mathcal{Q}_{\ell+1,\ell}(\underline{p}_{\ell+1}, \underline{p}_{\ell}|q) = Q_{\ell+1,\ell}(q^{p_{\ell+1,i}+i-1}, q^{-p_{\ell,i}-i+1}|q).$$

Corollary 4.1 *The following explicit expression for $\mathcal{Q}_{\ell+1,\ell}(\underline{p}_{\ell+1}, \underline{p}_{\ell}|q)$ holds*

$$\mathcal{Q}_{\ell+1,\ell}(\underline{p}_{\ell+1}, \underline{p}_{\ell}|q) = \frac{\prod_{i=1}^{\ell} \Theta(p_{\ell,i} - p_{\ell+1,i}) \Theta(p_{\ell+1,i+1} - p_{\ell,i})}{\prod_{i=1}^{\ell} (p_{\ell,i} - p_{\ell+1,i})_q! (p_{\ell+1,i+1} - p_{\ell,i})_q!},$$

where $\Theta(n) = 1$ when $n \geq 0$ and $\Theta(n) = 0$ otherwise.

Proposition 4.4 *For any $k = 1, \dots, \ell + 1$ the following intertwining relations hold*

$$\mathcal{H}_k^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) \mathcal{Q}_{\ell+1,\ell}(\underline{p}_{\ell+1}, \underline{p}_{\ell}|q) = \left\{ \tilde{\mathcal{H}}_{k-1}^{\mathfrak{gl}_{\ell}}(-\underline{p}_{\ell}) + \tilde{\mathcal{H}}_k^{\mathfrak{gl}_{\ell}}(-\underline{p}_{\ell}) \right\} \mathcal{Q}_{\ell+1,\ell}(\underline{p}_{\ell+1}, \underline{p}_{\ell}|q). \quad (4.6)$$

Proof: Follows from Proposition 4.3 \square

4.3 Pairing

Define a pairing:

$$\langle f, g \rangle_q = \oint_{y_1=0} \cdots \oint_{y_{\ell}=0} \prod_{i=1}^{\ell} \frac{dy_i}{2\pi i y_i} \Delta(y) f(y^{-1}) g(y), \quad (4.7)$$

where

$$\Delta(y) = \prod_{n=1}^{\infty} \prod_{i=1}^{\ell-1} \frac{1 - q^n}{1 - y_{i+1} y_i^{-1} q^{n-1}}, \quad f(y^{-1}) := f(y_1^{-1}, \dots, y_{\ell}^{-1}). \quad (4.8)$$

Proposition 4.5 *Hamiltonians $\mathcal{H}_r^{\mathfrak{gl}_{\ell}}(y)$ and $\tilde{\mathcal{H}}_r^{\mathfrak{gl}_{\ell}}(y)$ are adjoint with respect to the pairing (4.7)*

$$\langle f, \mathcal{H}_k^{\mathfrak{gl}_{\ell}} g \rangle_q = \langle \tilde{\mathcal{H}}_k^{\mathfrak{gl}_{\ell}} f, g \rangle_q, \quad k = 1, \dots, \ell.$$

Proof: Let us adopt the following notations

$$\mathcal{H}_r^{\mathfrak{gl}_{\ell}}(y) = \sum_{I_r} A_{I_r}(y) T_{I_r}, \quad \tilde{\mathcal{H}}_r^{\mathfrak{gl}_{\ell}}(y) = \sum_{I_r} B_{I_r}(y) T_{I_r},$$

where $T_{I_r} := T_{i_1} T_{i_2} \cdots T_{i_r}$. One should prove

$$\begin{aligned} & \oint_{y_1=0} \cdots \oint_{y_\ell=0} \prod_{i=1}^{\ell} \frac{dy_i}{2\pi i y_i} \Delta(y) f(y^{-1}) \sum_{I_r} T_{I_r} \cdot T_{I_r}^{-1} A_{I_r}(y) T_{I_r} g(y) \\ &= \oint_{y_1=0} \cdots \oint_{y_\ell=0} \prod_{i=1}^{\ell} \frac{dy_i}{2\pi i y_i} \Delta(y) \left(\sum_{I_r} T_{I_r}^{-1} A_{I_r}(y) T_{I_r} \cdot \frac{T_{I_r}^{-1} \Delta(y) T_{I_r}}{\Delta(y)} \cdot T_{I_r}^{-1} f(y^{-1}) \right) g(y). \end{aligned}$$

Then the proof is provided by the following Lemma \square

Lemma 4.1 *For any $I_r = (i_1 < i_2 < \dots < i_r) \subset \{1, 2, \dots, \ell\}$ the following relation holds.*

$$B_{I_r}(y) = \left(\Delta_{I_r}(y) \cdot T_{I_r}^{-1} A_{I_r}(y) T_{I_r} \right)^*,$$

where

$$\Delta_{I_r}(y) = (\Delta(y))^{-1} T_{I_r}^{-1} \Delta(y) T_{I_r},$$

for all $i \in I_r$. Where given a function $f(y)$ we define

$$f^*(y) := f(y^{-1}).$$

Proof: By direct calculation one derives

$$(T_{I_r}^{-1} A_{I_r}(y) T_{I_r})^* = \prod_{k=1}^r \left(1 - q^{-1} \frac{y_{i_k-1}}{y_{i_k}} \right)^{1-\delta_{i_k-i_{k-1}, 1}},$$

and

$$\Delta_{I_r}^*(y) = \prod_{k=1}^r \frac{\left(1 - \frac{y_{i_k}}{y_{i_k+1}} \right)^{1-\delta_{i_k+1-i_k, 1}}}{\left(1 - q^{-1} \frac{y_{i_k-1}}{y_{i_k}} \right)^{1-\delta_{i_k-i_{k-1}, 1}}},$$

where we set $i_0 := 0$, $i_{r+1} := \ell + 1$. In this way we obtain

$$(T_{I_r}^{-1} A_{I_r}(y) T_{I_r})^* \cdot \Delta_{I_r}^*(y) = Y_{i_1}^{1-\delta_{i_2-i_1, 1}} \cdots Y_{i_{r-1}}^{1-\delta_{i_r-i_{r-1}, 1}} \cdot Y_{i_r}^{1-\delta_{i_{r+1}-i_r, 1}}$$

\square

To construct recursive formulas for q -deformed Whittaker functions one should introduce a pairing on a functions defined on the lattice $\{y_i = q^{p_{\ell, i} + i - 1}; i = 1, \dots, \ell; p_{\ell, i} \in \mathbb{Z}\}$ with appropriate decay at infinities. Let us define the following analog of (4.7)

$$\langle f, g \rangle_{\text{lat}} = \sum_{\underline{p}_{\ell} \in \mathbb{Z}^{\ell}} \Delta'(\underline{p}_{\ell}) f(-\underline{p}_{\ell}) g(\underline{p}_{\ell}), \quad (4.9)$$

where

$$\Delta'(\underline{p}_{\ell}) = \prod_{i=1}^{\ell-1} \Theta(p_{\ell, i+1} - p_{\ell, i}) (p_{\ell, i+1} - p_{\ell, i})_q!. \quad (4.10)$$

The following Proposition can be easily proved by mimicking the proof of the Proposition 4.5.

Proposition 4.6 *Hamiltonians $\mathcal{H}_r^{\text{gl}_{\ell}}(\underline{p}_{\ell})$ and $\tilde{\mathcal{H}}_r^{\text{gl}_{\ell}}(\underline{p}_{\ell})$ are adjoint with respect to the pairing (4.9)*

$$\langle f, \mathcal{H}_k^{\text{gl}_{\ell}} g \rangle_{\text{lat}} = \langle \tilde{\mathcal{H}}_k^{\text{gl}_{\ell}} f, g \rangle_{\text{lat}}, \quad k = 1, \dots, \ell. \quad (4.11)$$

4.4 Proof of Theorem 2.1

Now we are ready to prove Theorem 2.1. We use recursion over the rank of \mathfrak{gl}_k . Set $\Psi_{\lambda_1}^{\mathfrak{gl}_1}(p_{11}) = q^{l_1 p_{11}}$ and assume that

$$\begin{aligned} \mathcal{H}_r^{\mathfrak{gl}_\ell}(\underline{p}_\ell) \cdot \Psi_{l_1, \dots, l_\ell}^{\mathfrak{gl}_\ell}(\underline{p}_\ell) &= \chi_r^{\mathfrak{gl}_\ell}(q^{\sum_i l_i E_{ii}}) \Psi_{l_1, \dots, l_\ell}^{\mathfrak{gl}_\ell}(\underline{p}_\ell), \\ \chi_r^{\mathfrak{gl}_\ell}(q^{\sum_i l_i E_{ii}}) &= \sum_{I_r^{(\ell)}} z_{i_1} z_{i_2} \cdots z_{i_r}, \quad z_i = q^{l_i}, \end{aligned} \quad (4.12)$$

where $I_r^{(\ell)} = \{i_1 < i_2 < \dots < i_r\} \in (1, 2, \dots, \ell)$.

Let us define the function $\Psi_{l_1, \dots, l_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1})$ as follows

$$\begin{aligned} \Psi_{l_1, \dots, l_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) &= \sum_{\underline{p}_\ell \in \mathbb{Z}^\ell} \Delta'(\underline{p}_\ell) \mathcal{Q}_{\ell+1, \ell}(\underline{p}_{\ell+1}, \underline{p}_\ell) \\ &\quad \cdot q^{l_{\ell+1}(\sum_{i=1}^{\ell+1} p_{\ell+1, i} - \sum_{i=1}^\ell p_{\ell, i})} \Psi_{l_1, \dots, l_\ell}^{\mathfrak{gl}_\ell}(\underline{p}_\ell), \end{aligned} \quad (4.13)$$

where

$$\mathcal{Q}_{\ell+1, \ell}(\underline{p}_{\ell+1}, \underline{p}_\ell) = \frac{\prod_{i=1}^\ell \Theta(p_{\ell, i} - p_{\ell+1, i}) \Theta(p_{\ell+1, i+1} - p_{\ell, i})}{\prod_{i=1}^\ell (p_{\ell, i} - p_{\ell+1, i})_q! (p_{\ell+1, i+1} - p_{\ell, i})_q!},$$

and

$$\Delta'(\underline{p}_\ell) = \prod_{i=1}^{\ell-1} \Theta(p_{\ell, i+1} - p_{\ell, i}) (p_{\ell, i+1} - p_{\ell, i})_q!.$$

One should verify the relations:

$$\begin{aligned} \mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) \cdot \Psi_{l_1, \dots, l_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) &= \chi_r^{\mathfrak{gl}_{\ell+1}}(q^{\sum_i l_i E_{ii}}) \Psi_{l_1, \dots, l_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}), \\ \chi_r^{\mathfrak{gl}_{\ell+1}}(q^{\sum_i l_i E_{ii}}) &= \sum_{I_r^{(\ell+1)}} z_{i_1} z_{i_2} \cdots z_{i_r}, \quad z_i = q^{l_i}, \end{aligned} \quad (4.14)$$

where $I_r^{(\ell+1)} = \{i_1 < i_2 < \dots < i_r\} \in (1, 2, \dots, \ell+1)$.

Applying Hamiltonians $\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1})$ to (4.13) and using intertwining relation given in Proposition 4.4 one can obtains

$$\begin{aligned} &\mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) \mathcal{Q}_{\ell+1, \ell}(\underline{p}_{\ell+1}, \underline{p}_\ell) q^{l_{\ell+1}(\sum_i p_{\ell+1, i} - \sum_i p_{\ell, i})} = \\ &= \left\{ q^{l_{\ell+1}} \tilde{\mathcal{H}}_{r-1}^{\mathfrak{gl}_\ell}(-\underline{p}_\ell) + \tilde{\mathcal{H}}_r^{\mathfrak{gl}_\ell}(-\underline{p}_\ell) \right\} \mathcal{Q}_{\ell+1, \ell}(\underline{p}_{\ell+1}, \underline{p}_\ell) q^{l_{\ell+1}(\sum_i p_{\ell+1, i} - \sum_i p_{\ell, i})}. \end{aligned}$$

Now using (4.11), one obtains

$$\begin{aligned}
& \mathcal{H}_r^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) \Psi_{l_1, \dots, l_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) = \\
& = \sum_{\underline{p}_\ell \in \mathbb{Z}^\ell} \Delta(\underline{p}_\ell) \left(\mathcal{H}_r^{\mathfrak{gl}_\ell}(\underline{p}_\ell) \mathcal{Q}_{\ell+1, \ell}(\underline{p}_{\ell+1}, \underline{p}_\ell | q) q^{l_{\ell+1}(\sum_i p_{\ell+1, i} - \sum_i p_{\ell, i})} \right) \Psi_{l_1, \dots, l_\ell}^{\mathfrak{gl}_\ell}(\underline{p}_\ell) \\
& = \sum_{\underline{p}_\ell \in \mathbb{Z}^\ell} \Delta(\underline{p}_\ell) (q^{l_{\ell+1}} \tilde{\mathcal{H}}_{r-1}^{\mathfrak{gl}_\ell}(-\underline{p}_\ell) + \tilde{\mathcal{H}}_r^{\mathfrak{gl}_\ell}(-\underline{p}_\ell)) \left(\mathcal{Q}_{\ell+1, \ell}(\underline{p}_{\ell+1}, \underline{p}_\ell) q^{l_{\ell+1}(\sum_i p_{\ell+1, i} - \sum_i p_{\ell, i})} \right) \Psi_{l_1, \dots, l_\ell}^{\mathfrak{gl}_\ell}(\underline{p}_\ell) \\
& = \sum_{\underline{p}_\ell \in \mathbb{Z}^\ell} \Delta(\underline{p}_\ell) \left(\mathcal{Q}_{\ell+1, \ell}(\underline{p}_{\ell+1}, \underline{p}_\ell) q^{l_{\ell+1}(\sum_i p_{\ell+1, i} - \sum_i p_{\ell, i})} \right) (q^{l_{\ell+1}} \mathcal{H}_{r-1}^{\mathfrak{gl}_\ell}(\underline{p}_\ell) + \mathcal{H}_r^{\mathfrak{gl}_\ell}(\underline{p}_\ell)) \Psi_{l_1, \dots, l_\ell}^{\mathfrak{gl}_\ell}(\underline{p}_\ell) \\
& = \left(q^{l_{\ell+1}} \sum_{I_{r-1}^{(\ell)}} \prod_{i \in I_{r-1}^{(\ell)}} q^{l_i} + \sum_{I_r^{(\ell)}} \prod_{i \in I_r^{(\ell)}} q^{l_i} \right) \Psi_{l_1, \dots, l_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}) = \left(\sum_{I_r^{(\ell+1)}} \prod_{i \in I_r^{(\ell+1)}} q^{l_i} \right) \Psi_{l_1, \dots, l_{\ell+1}}^{\mathfrak{gl}_{\ell+1}}(\underline{p}_{\ell+1}),
\end{aligned}$$

where $I_r^{(\ell)} = \{i_1 < i_2 < \dots < i_r\} \in (1, 2, \dots, \ell)$ and $I_r^{(\ell+1)} = \{i_1 < i_2 < \dots < i_r\} \in (1, 2, \dots, \ell+1)$. In the last equality above we use the following relation.

$$\chi_r^{\mathfrak{gl}_{\ell+1}}(\underline{z}) = z_{\ell+1} \chi_{r-1}^{\mathfrak{gl}_\ell}(\underline{z}') + \chi_r^{\mathfrak{gl}_\ell}(\underline{z}'),$$

where $\underline{z}' = (z_1, z_2, \dots, z_\ell)$ for $z_i = q^{l_i}$. This completes the proof of Theorem 2.1 \square

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